Numerically Solving Parametric Families of High-Dimensional Kolmogorov Partial Differential Equations via Deep Learning

Julius Berner, Markus Dablander, Philipp Grohs

University of Vienna, University of Oxford

Kolmogorov PDEs

- **Parametric Kolmogorov PDEs** are partial differential equations of the form

$$\frac{\partial u_\gamma}{\partial t} = \frac{1}{2}\mathrm{Trace}\big(\sigma_\gamma[\sigma_\gamma]^* \nabla_x^2 u_\gamma\big) + \langle \mu_\gamma, \nabla_x u_\gamma \rangle, \quad u_\gamma(x,0) = \varphi_\gamma(x),$$

whereby the initial condition and the coefficient maps

$$\varphi_\gamma : \mathbb{R}^d \to \mathbb{R}, \quad \sigma_\gamma : \mathbb{R}^d \to \mathbb{R}^{d\times d}, \quad \mu_\gamma : \mathbb{R}^d \to \mathbb{R}^d$$

are implicitly determined by a **real parameter vector $\gamma \in D$**.

- **Relevance:** Kolmogorov PDEs frequently appear in physics (heat equation) and financial engineering (Black-Scholes model).
- **Challenges:** Kolmogorov PDEs can generally not be solved explicitly. Furthermore, standard numerical solution algorithms suffer from the **curse of dimensionality**, meaning that their computational cost grows exponentially in the spatial dimension d.

Contribution: Parametric PDE Solution via Deep Learning

- **Novel Solution Algorithm:** We introduce a new deep learning algorithm which makes it possible to train a single deep network

$$\Phi \colon D \times [v,w]^d \times [0,T] \to \mathbb{R}$$

to approximate the **parametric Kolmogorov PDE solution map**

$$\bar{u} : D \times [v,w]^d \times [0,T] \to \mathbb{R}, \quad (\gamma, x, t) \mapsto \bar{u}(\gamma, x, t) := u_\gamma(x,t),$$

of a family of γ-parametrized Kolmogorov PDEs.

- **Successful Experiments:** We propose a new **Multilevel architecture** for Φ and empirically confirm the functionality of our technique for challenging examples from physics and computational finance.
- **Theoretical Guarantees:** We investigate the approximation- and generalization errors of our method and show that **the proposed algorithm does not suffer from the curse of dimensionality** in various important cases.
- **Novel Parametric Analysis:** The approximation $\Phi \approx \bar{u}$ allows for sensitivity analysis, model calibration, and uncertainty quantification.

Algorithm

- **Key Idea:** To describe the parametric PDE solution map $\bar{\boldsymbol{u}}$ as the regression function of a **supervised statistical learning problem** and then use **simulated training data** to learn $\bar{u}$ via deep learning.
- **Predictor and Target:** The predictor Λ is uniformly distributed,

$$\Lambda := (\Gamma, X, \mathcal{T}) \sim \mathcal{U}(D \times [v,w]^d \times [0,T]),$$

and the target variable S_Λ is defined as the value of the solution process $(S_{\Gamma,X,t})_{t\geq 0}$ of the Γ-parametrized **stochastic differential equation**

$$dS_{\Gamma,X,t} = \mu_\Gamma(S_{\Gamma,X,t})dt + \sigma_\Gamma(S_{\Gamma,X,t})dB_t, \quad S_{\Gamma,X,0} = X,$$

at the random stopping time $t = \mathcal{T}$. We can easily simulate i.i.d. samples of S_Λ via the **Euler-Maruyama scheme**.

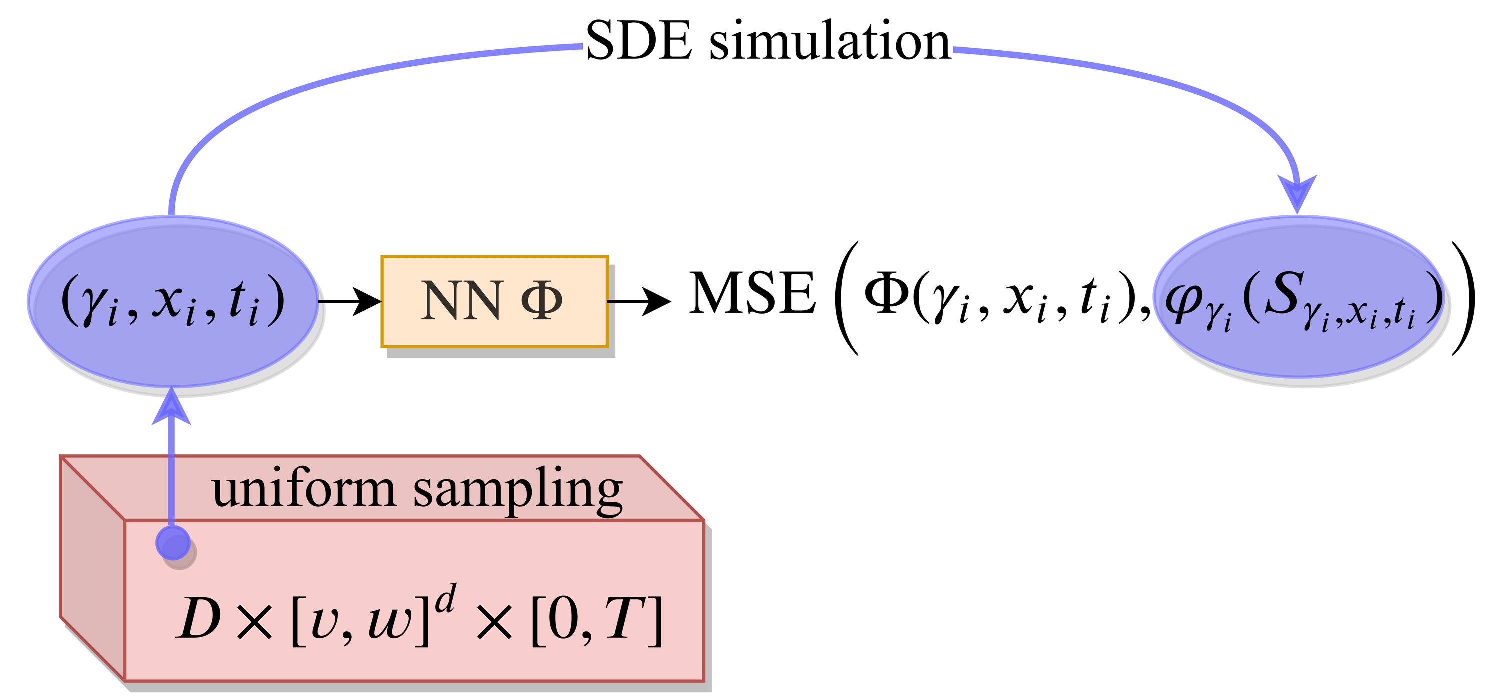
- **Supervised Learning Problem with Simulated Data:**

Theorem (Main Learning Problem)

For suitable regularity assumptions, the parametric PDE solution map $\bar{u}$ is the unique minimizer of the statistical learning problem

$$\min_f \mathbb{E}\Big[\big(f(\Lambda) - \varphi_\Gamma(S_\Lambda)\big)^2\Big].$$

The proof of the above statement relies on an application of the **Feynman-Kac formula**, which links the SDE solution to the PDE solution via $\mathbb{E}[\varphi_\gamma(S_{\gamma,x,t})] = u_\gamma(x,t)$. The theorem delivers a feasible supervised learning problem with solution $\bar{u}$ for which an **endless stream of i.i.d. training data points** can be simulated.



Numerical Results

- **Basket Put Option Pricing:** Consider the following setting for $d = 3$ on $(x,t) \in [9,10]^3 \times [0,1]$ with matrices $\gamma_{\sigma,1}, \ldots, \gamma_{\sigma,4}, \gamma_{\mu,1} \in [0.1, 0.6]^{3\times 3}$, vector $\gamma_{\mu,2} \in [0.1, 0.6]^3$ and scalar $\gamma_\varphi \in [10, 12]$:

$$\begin{aligned}
\sigma_\gamma &: \mathbb{R}^3 \to \mathbb{R}^{3\times 3}, & \sigma_\gamma(x) &= [\gamma_{\sigma,1}x | \gamma_{\sigma,2}x | \gamma_{\sigma,3}x] + \gamma_{\sigma,4}, \\
\mu_\gamma &: \mathbb{R}^3 \to \mathbb{R}^3, & \mu_\gamma(x) &= \gamma_{\mu,1}x + \gamma_{\mu,2}, \\
\varphi_\gamma &: \mathbb{R}^3 \to \mathbb{R}, & \varphi_\gamma(x) &= \max\big\{\gamma_\varphi - \tfrac{1}{3}(x_1 + x_2 + x_3), 0\big\}.
\end{aligned}$$

The corresponding Kolmogorov PDE describes the evolution of a Basket put option price in a parametric multidimensional Black-Scholes model with $d = 3$ potentially highly correlated assets. Our method allows a deep network Φ to efficiently converge to the solution map $\bar{u}$.

Gradient Steps	Time [s]	Error $\approx \lVert(\Phi - \bar{u})/(1 + \lvert\bar{u}\rvert)\rVert_{L^1}$
0	0 ± 0	0.7912 ± 0.0276
12k	2434 ± 28	0.0062 ± 0.0009
20k	4162 ± 154	0.0046 ± 0.0007
28k	6024 ± 463	0.0039 ± 0.0001

- **Additional Experiments:** (1) classical parametric Black-Scholes model for single option pricing, (2) high-dimensional parametric heat equations with parabolic and Gaussian initial conditions.
- **Multilevel Architecture:**